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Relativistic unipolar induction

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Abstract. The stationary electromagnetic field produced by a rapidly rotating, magnetised and conducting sphere is discussed. The field is determined by solving Maxwell's equations for an accelerating medium and reduces in the non-relativistic limit to the expected unipolar induction field of a slowly rotating spherical magnet.

1. Introduction

The classical electrodynamics of continuous media provides a successful description of many phenomena involving matter and electromagnetism. If the motions of the media are prescribed along with appropriate constitutive relations and boundary conditions then Maxwell's equations provide a closed system for many problems of interest. Among some of the earliest applications [1, 2 and further references therein] one may include the problem of determining the electromagnetic field of a rotating spherical conductor with permanent magnetisation. Such a system underlies the modelling of a wide variety of diverse phenomena ranging from the Earth's geomagnetism [3] to the description of pulsar dynamics [4]. It may also be of relevance for the description of extended relativistic [5] models of certain elementary particles. It is somewhat surprising to discover that this problem has generated considerable debate [6] in the not-too-distant scientific literature. The reason for this appears twofold. Firstly, early investigators were often imprecise in posing the problem and fixing the frame of reference for their subsequent solutions. Secondly, there seemed to be a reluctance to use the covariant formulation of Maxwell's equations. Thus one finds certain authors excusing themselves for using 'Minkowski's electrodynamics of moving media' (which establish the relation between electromagnetic fields measured in different inertial frames) in a problem dealing with fields in a non-inertial frame. The reasons given were that such an approximation neglected effects that were deemed irrelevant in the context under discussion. However, to our knowledge such effects have never been sought or discussed in the context of the unipolar induction of a rapidly rotating spherical magnet.

Our aim in resurrecting this classical problem is to demonstrate that its covariant spacetime formulation rectifies and clarifies a number of conceptual errors that plagued the early treatments.

We recognise that the electromagnetic fields for this problem must depend on the nature of the choice of magnetisation tensor in the sphere. Such a tensor could in general depend on the acceleration of the spinning sphere and in any case will depend

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on the physics of extended matter as the spin rate becomes relativistic. Furthermore the type of magnetisation adopted, the induced currents in the medium and the interior and exterior fields must all be consistent with the covariant Maxwell equations. In the absence of any persuasive properties for a relativistic magnetisation in an accelerating medium we approach this problem by specifying the types of current permitted in the rotating sphere and seeking the simplest magnetisation consistent with these that reduces in the static limit to a uniform magnetisation. In order to make contact with earlier work we suppose that the sphere sustains a convective 4-current proportional to the 4-velocity of each rotating element together with a possible ohmic current proportional to the local electric field in the corotating frame of the medium. Such a minimal assumption enables us to find a simple self-consistent magnetisation field with the required limiting properties. We then seek stationary fields that render the ohmic currents zero. It is customary to describe such a solution as appropriate to an infinitely conducting medium although we prefer to regard it as the stationary limit of a solution involving a finite conductivity. Thus no recourse is made to inertial Lorentz transformations; one simply solves Maxwell's equations with well posed boundary conditions in the context of a (flat) spacetime description of a rotating magnetised sphere. We present an exact solution that reduces to the expected field configuration in the low rotation speed limit but fully accommodates the effects of relativistic speeds of rotation.

Although we are not concerned here with pursuing this simple model into astrophysical domains one might regard it as a relativistic version of the Julian–Goldreich [7] symmetric pulsar model. It would be of interest to generalise our techniques to a non-axially symmetric configuration containing a magnetosphere in order to see if our relativistic generalisation could have non-trivial implications in a more realistic situation.

2. Covariant formulation

We seek the fields generated by a rotating magnetised sphere. Let g denote the metric tensor field on spacetime which we shall take to be Minkowskian. A non-vanishing future-pointing timelike vector field v on a region of spacetime can be normalised so that:

$$g(v, v) = -1 \tag{2.1}$$

and it may be associated with an observer frame. Then if F is the closed Maxwell 2-form ($\mathrm{d}F = 0$) it admits an orthogonal decomposition with respect to any observer field v according to:

$$F = \boldsymbol{e} \wedge \tilde{v} + c * (\boldsymbol{b} \wedge \tilde{v}) \tag{2.2}$$

where the 1-forms $\boldsymbol{e}$ and $\boldsymbol{b}$ satisfy

$$i_v \boldsymbol{e} = 0 \tag{2.3}$$

$$i_v \boldsymbol{b} = 0 \tag{2.4}$$

and are defined as the electric and magnetic induction 1-forms with respect to such an observer field. i_v here denotes the interior or contraction operator on forms [8] and $c^2 = 1/\varepsilon_0\mu_0$ in terms of the permittivity ε_0 and permeability μ_0 of the vacuum. The tilde over a form associates it with the corresponding vector field. Thus given v, $\tilde{v}$ is defined by

$$\tilde{v}(X) = g(v, X) \tag{2.5}$$

for all vector fields X. The classical electric and magnetic induction vector fields correspond to $\tilde{\boldsymbol{e}}$ and $\tilde{\boldsymbol{b}}$. The (linear) map $*$ is the Hodge map defined with respect to g, taking forms to their Hodge dual. Given F and v, it follows that

$$\boldsymbol{e} = i_v F \tag{2.6}$$

$$\boldsymbol{b} = -i_v * F. \tag{2.7}$$

In a polarisable medium with permanent magnetisation it is customary to incorporate certain source of F into a 2-form G. Then if j denotes the remaining current 1-form the remaining Maxwell equation becomes

$$\delta G = j \tag{2.8}$$

where $\delta = *^{-1}\mathrm{d}*$. The displacement 1-form $\boldsymbol{d}$ and the magnetic 1-form $\boldsymbol{h}$, determined by v follow from:

$$G = \boldsymbol{d} \wedge \tilde{v} + \frac{1}{c} * (\boldsymbol{h} \wedge \tilde{v}) \tag{2.9}$$

with $i_v \boldsymbol{d} = i_v \boldsymbol{h} = 0$. Similarly the scalar volume charge density $\rho_{(v)}$ and 3-current 2-form $J_{(v)}$ determined by v are given by:

$$j = -\rho_{(v)}\tilde{v} + *(J_{(v)} \wedge \tilde{v}). \tag{2.10}$$

We call the difference between G and $\varepsilon_0 F$ the polarisation 2-form Π according to:

$$G = \varepsilon_0 F + \Pi. \tag{2.11}$$

A polarisable medium I will be modelled on a 4-chain on spacetime on which is defined a timelike vector field u describing its state of motion. For our purpose we may suppose that u has been normalised ($g(u, u) = -1$). Then the polarisation 1-form $\boldsymbol{p}$ and magnetisation 1-form $\boldsymbol{m}$ are defined with respect to u by:

$$\Pi = \boldsymbol{p} \wedge \tilde{u} - \frac{1}{c} * (\boldsymbol{m} \wedge \tilde{u}) \tag{2.12}$$

with $i_u \boldsymbol{p} = i_u \boldsymbol{m} = 0$. If the medium contains proper charge density $\rho_{(u)}$ then j will contain the (convective) current 1-form $\rho_{(u)}\tilde{u}$. The medium is (isotropically) electrically conducting if the 4-chain with support on region I of spacetime has a scalar $\sigma_\Omega > 0$ (the conductivity) associated with it such that j contains as a contribution the ohmic current

$$j_\Omega = -\sigma_\Omega i_u F. \tag{2.13}$$

A medium having an interface with (say) the vacuum will be modelled by a four-dimensional region with a timelike 3-chain giving the history of the interface as the hypersurface

$$f = 0 \tag{2.14}$$

in terms of some smooth function f on spacetime. Hence $\widetilde{\mathrm{d}f}$ will be spacelike. We shall assume that for region I, $f < 0$ while $f > 0$ will be denoted as a vacuum region II.

Thus a simple medium is characterised by the function f that defines the domain $f < 0$ of a vector field u, together with the 1-forms $(\boldsymbol{p}, \boldsymbol{m})$ and the scalar σ_Ω on spacetime. Given f, u and σ_Ω we seek solutions to Maxwell's equations

$$\mathrm{d}F^{\mathrm{I}} = 0 \tag{2.15}$$

$$\delta G^{\mathrm{I}} = -\rho_{(u)}\tilde{u} - \sigma_\Omega i_u F^{\mathrm{I}} \tag{2.16}$$

that can be matched consistently to vacuum solutions of the equations:

$$\mathrm{d}F^{\mathrm{II}} = 0 \tag{2.17}$$

$$\delta G^{\mathrm{II}} = 0 \tag{2.18}$$

where $G^{\mathrm{II}} = \varepsilon_0 F^{\mathrm{II}}$.

3. Invariant boundary conditions

Since $f = 0$ defines a discontinuous interface between two media we must deal with discontinuous 2-forms on spacetime and admit a distributional current 1-form having support on the 3-chain defined by $f = 0$. If δ_f denotes a hypersurface-Dirac distribution then such a current may be written $j_\Delta \delta_f$ where j_Δ is regular on $f = 0$. Denoting the discontinuity of any form as $[F] \equiv (F^{\mathrm{I}} - F^{\mathrm{II}})$ evaluated at $f = 0$, it follows from (2.15)–(2.18) that

$$[F] \wedge \mathrm{d}f|_{f=0} = 0 \tag{3.1}$$

$$[*G] \wedge \mathrm{d}f|_{f=0} = *j_\Delta. \tag{3.2}$$

Since $[*G] \wedge \mathrm{d}f = -*i_{\widetilde{\mathrm{d}f}}[G]$, equation (3.2) may alternatively be written as

$$i_{\widetilde{\mathrm{d}f}}[G] = -j_\Delta. \tag{3.3}$$

To reformulate the above equation in terms of electric and magnetic fields one must choose (in addition to the vector field u defining the motion of the medium) an observer field v with which to refer the 1-forms $\boldsymbol{e}$, $\boldsymbol{b}$, $\boldsymbol{d}$ and $\boldsymbol{h}$. (There is no reason why the frames so chosen for region I and II should coincide.) Contrary to assertions that are sometimes found in the literature, the (four-dimensional) Maxwell's equations are perfectly capable of describing accelerating matter and can accommodate fields defined with respect to accelerating observers. For some problems a comoving field description with $v = u$ may be appropriate but if $\mathrm{d}\tilde{v} \neq 0$ then projection of F into comoving fields will entail 'non-inertial' terms simulating current sources in the $(3+1)$-form of Maxwell's equations. In the following we shall work with an inertial v ($\mathrm{d}v = 0$) which defines what we shall call a laboratory frame.

We suppose our sphere to have no proper polarisation and take $\boldsymbol{p} = 0$. Since it is not our concern to worry about elastic properties of the medium we shall also suppose our sphere to be an idealised 'rigid' rotator with a constant angular speed ω about a fixed spatial axis in the laboratory frame. Furthermore we demand that when $\omega = 0$ the sphere possess a constant uniform magnetisation directed along the axis of rotation. In this configuration we naturally demand the sphere to be electrically neutral and that the only magnetic fields in this frame are those produced by the magnetisation. Thus the sphere contains no surface and volume charge density when $\omega = 0$. The imposition of these conditions on our problem together with the regularity conditions on the fields at spatial infinity are necessary to pin down the solution of the rotating sphere. In particular our solution is determined by our choice of admissible currents $(j, \mathrm{d} * \Pi)^{\mathrm{I}}$ and the condition that the solution reduces in the $\omega = 0$ limit to the inertial field configuration described above. Furthermore our solution implies the existence of free charge in the interior of the sphere having a non-zero net flux (of $*G^{\mathrm{I}}$) over its surface. However, this will be exactly balanced by an integrated surface charge induced by the discontinuity of $*G$ across the hypersurface $f = 0$. The latter depends on the structure of magnetisation form Π as well as the condition that the surface 3-currents vanish as ω tends to zero.

We impose two spacetime symmetries on our solution. We are interested in axially symmetric, stationary, inertial-frame field configurations. Since the presence of a finite conductivity provides a natural relaxation timescale $(\varepsilon_0/\sigma_\Omega)$ for time-dependent solutions, the stationary solutions are expected only in the infinite conductivity limit or in the asymptotic time limit of a transient solution with finite σ_Ω. Thus we seek a stationary inertial field satisfying

$$i_u F^{\mathrm{I}} = 0. \tag{3.4}$$

If X generates a spacetime symmetry axis we also require that

$$\mathscr{L}_X \Pi = 0 \tag{3.5}$$

$$\mathscr{L}_X F = 0 \tag{3.6}$$

where $\mathscr{L}_X$ denotes the Lie derivative with respect to X.

4. Interior fields

Let (t, x, y, z) be standard inertial coordinates for Minkowskian spacetime for which

$$g = -\mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z. \tag{4.1}$$

We adopt units in which $c = 1$ henceforth and suppose $x = y = z = 0$ locates the world line of the centre of the sphere. In standard spherical polar coordinates (t, r, θ, ϕ)

$$g = -\mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}r \otimes \mathrm{d}r + r^2\, \mathrm{d}\theta \otimes \mathrm{d}\theta + r^2 \sin^2\theta\, \mathrm{d}\phi \otimes \mathrm{d}\phi \tag{4.2}$$

and we suppose the sphere rotates about the z axis where $z = r\cos\theta$. We define the inertial electric and magnetic fields with respect to $v = \partial/\partial t$ with $\tilde{v} = -\mathrm{d}t$. In terms of the polar chart the interior of the sphere is defined by $f < 0$ where

$$f = r - a \tag{4.3}$$

for some constant a (the radius of the sphere). The 'rigid' rotation corresponds to the vector field

$$u = \gamma(\partial_t + \omega \partial_\phi) \tag{4.4}$$

where

$$\gamma^2(r, \theta) = (1 - \omega^2 r^2 \sin^2\theta)^{-1} \tag{4.5}$$

and the constant ω is such that:

$$\omega^2 < 1/(r^2 \sin^2\theta) \qquad \text{for } 0 \leqslant r \leqslant a,\ 0 \leqslant \theta \leqslant \pi.$$

The integral curves of u may be identified with the world lines of the elements composing the rotating medium. Since $\partial/\partial z$ is in the direction of the axis of rotation we endow the sphere with a magnetisation described by the 2-form

$$\Pi = -*(\boldsymbol{m} \wedge \tilde{u}) \tag{4.6}$$

where $\boldsymbol{m}$ is a spacelike 1-form on spacetime with the properties

$$\mathscr{L}_{\partial_\phi} \boldsymbol{m} = 0 \tag{4.7}$$

$$i_u \boldsymbol{m} = 0. \tag{4.8}$$

As $\tilde{u} = \gamma(-\mathrm{d}t + \omega r^2 \sin^2\theta \, \mathrm{d}\phi)$ we accommodate our symmetry requirements by looking for solutions in which the magnetisation is proportional to $m_0 \, \mathrm{d}z$ for a specified constant m_0. Thus our fields will be parametrised by the radius of the sphere a, the angular speed ω and the magnetisation strength m_0.

In terms of an orthonormal coframe basis $e^0 = \mathrm{d}t$, $e^1 = \mathrm{d}r$, $e^2 = r\,\mathrm{d}\theta$, $e^3 = r \sin\theta \, \mathrm{d}\phi$, the equations (3.4), (2.15) and (2.16) are satisfied by

$$F^{\mathrm{I}} = \gamma b_0 * (\mathrm{d}z \wedge \tilde{u})$$

or

$$F^{\mathrm{I}} = -\gamma^2 b_0(\omega r \sin^2\theta e^0 \wedge e^1 + \omega r \sin\theta \cos\theta e^0 \wedge e^2 + \cos\theta e^2 \wedge e^3 + \sin\theta e^1 \wedge e^3) \tag{4.9}$$

$$G^{\mathrm{I}} = \varepsilon_0 F^{\mathrm{I}} - \gamma m_0 * (\mathrm{d}z \wedge \tilde{u}) \tag{4.10}$$

for some constant b_0 to be determined.

The modulation of the 2-form $*(\mathrm{d}z \wedge \tilde{u})$ by the particular function γ of $r \sin\theta$ follows from the requirement that δG^{I} be proportional to $\tilde{u}$:

$$\delta F^{\mathrm{I}} = 2\gamma^3 \omega b_0 \tilde{u} \tag{4.11}$$

and

$$\delta(m_0 * (\mathrm{d}z \wedge \gamma \tilde{u})) = 2\gamma^3 \omega m_0 \tilde{u} \tag{4.12}$$

$$\begin{aligned} \delta G^{\mathrm{I}} &= 2\gamma^3 \omega(\varepsilon_0 b_0 - m_0)\tilde{u} \\ &= \rho_{(u)} \tilde{u}. \end{aligned} \tag{4.13}$$

Thus there appears a comoving volume charge density

$$\rho_{(u)} = 2\gamma^3 \omega(\varepsilon_0 b_0 - m_0) \tag{4.14}$$

in the sphere. The value of the charge density in the inertial frame is then given by

$$\begin{aligned} \rho_{(\partial_t)} &= \rho_{(u)} \tilde{u}(\partial_t) \\ &= 2(\varepsilon_0 b_0 - m_0)\gamma^4 \omega. \end{aligned} \tag{4.15}$$

This is accompanied by an azimuthal inertial 3-current density 2-form

$$J_{(\partial_t)} = \rho_{(\partial_t)} \omega r^2 \sin\theta \, \mathrm{d}r \wedge \mathrm{d}\theta. \tag{4.16}$$

From (4.11) and (4.14) we may compute the total free (volume) charge inside the sphere by

$$\int_{s^2} *G^{\mathrm{I}} = \frac{4\pi}{\omega^2}(\varepsilon_0 b_0 - m_0)\left(-\omega a + \frac{1}{\sqrt{1-\omega^2 a^2}} \sin^{-1} \omega a\right). \tag{4.17}$$

To fix b_0 we must next compute the vacuum fields outside the sphere.

5. Exterior fields

We now look for a stationary vacuum Maxwell field in the region II satisfying $\mathcal{L}_v F^{\mathrm{II}} = 0$, where $v = \partial/\partial t$ in our inertial chart, for the region II. In terms of 1-forms $\boldsymbol{e}^{\mathrm{II}}$ and $\boldsymbol{b}^{\mathrm{II}}$ with $i_v \boldsymbol{e}^{\mathrm{II}} = i_v \boldsymbol{b}^{\mathrm{II}} = 0$ we may write

$$F^{\mathrm{II}} = \boldsymbol{e}^{\mathrm{II}} \wedge \tilde{v} + *(\boldsymbol{b}^{\mathrm{II}} \wedge \tilde{v}) \tag{5.1}$$

The 1-forms $\boldsymbol{e}^{\mathrm{II}}$ and $\boldsymbol{b}^{\mathrm{II}}$ must obey

$$\mathrm{d}\boldsymbol{e}^{\mathrm{II}}=0 \tag{5.2}$$

$$\mathrm{d}\hat{*}\boldsymbol{e}^{\mathrm{II}}=0 \tag{5.3}$$

$$\mathrm{d}\boldsymbol{b}^{\mathrm{II}}=0 \tag{5.4}$$

$$\mathrm{d}\hat{*}\boldsymbol{b}^{\mathrm{II}}=0 \tag{5.5}$$

where the three-dimensional Hodge dual $\hat{*}$ is defined by

$$*1=\tilde{v}\wedge\hat{*}1. \tag{5.6}$$

Hence we have

$$\boldsymbol{e}^{\mathrm{II}}=-\mathrm{d}\Phi_{\mathrm{E}}^{\mathrm{II}} \tag{5.7}$$

$$\boldsymbol{b}^{\mathrm{II}}=-\mathrm{d}\Phi_{\mathrm{M}}^{\mathrm{II}} \tag{5.8}$$

for time-independent harmonic 0-forms $\Phi_{\mathrm{E}}^{\mathrm{II}}$ and $\Phi_{\mathrm{M}}^{\mathrm{II}}$. Adopting the symmetry conditions

$$\mathscr{L}_{\partial/\partial\phi}\Phi_{\mathrm{E}}^{\mathrm{II}}=\mathscr{L}_{\partial/\partial\phi}\Phi_{\mathrm{M}}^{\mathrm{II}}=0 \tag{5.9}$$

we have for $r\geqslant a$, $0\leqslant\theta\leqslant\pi$, $0\leqslant\phi<2\pi$

$$\Phi_{\mathrm{E}}^{\mathrm{II}}(r,\theta)=\sum_{j=1}^{\infty}\frac{\alpha_j}{r^{j+1}}P_j(\cos\theta) \tag{5.10}$$

$$\Phi_{\mathrm{M}}^{\mathrm{II}}(r,\theta)=\sum_{j=0}^{\infty}\frac{\beta_j}{r^{j+1}}P_j(\cos\theta) \tag{5.11}$$

in terms of the Legendre polynomials P_j. Exclusion of $j=0$ in (5.10) incorporates our charge-neutrality condition for the magnetised sphere. F^{II} can now be computed in terms of the numbers $\{\alpha_j\}$ and $\{\beta_j\}$, and compared with F^{I} at $f=0$. The condition

$$[F]\wedge\mathrm{d}r\big|_{r=a}=0 \tag{5.12}$$

then fixes the coefficients to be

$$\alpha_j=\frac{2j+1}{2}a^{j+1}\int_{-1}^{1}\Phi_{\mathrm{E}}^{\mathrm{I}}(a,\theta)P_j(\mu)\,\mathrm{d}\mu \tag{5.13}$$

$$\beta_j=\frac{2j+1}{2}\frac{a^{j+2}}{j+1}\int_{-1}^{1}b_r^{\mathrm{I}}(a,\theta)P_j(\mu)\,\mathrm{d}\mu \tag{5.14}$$

where

$$\Phi_{\mathrm{E}}^{\mathrm{I}}=\frac{b_0}{\omega}\ln\gamma(a,\theta) \tag{5.15}$$

and

$$b_r^{\mathrm{I}}=b_0\cos\theta\gamma^2(a,\theta)$$

$$\mu\equiv\cos\theta. \tag{5.16}$$

It follows from (5.13) and (5.14) that $\alpha_j=0$ for j odd and $\beta_j=0$ for j even.

Thus the exterior solution is specified in terms of the constants b_0, ω and a, as a multiple series starting with an electric quadrupole and a magnetic dipole field in the inertial frame.

6. Surface current

We finally compute the surface current and fix the constant b_0 by imposing the condition that when $\omega=0$, the surface of the sphere in the inertial frame carries no surface 3-current. We impose (3.3) with

$$G^{\mathrm{II}}=\varepsilon_0 F^{\mathrm{II}}. \tag{6.1}$$

From (4.10) and (6.1)

$$[G]=\varepsilon_0[F]-\gamma m_0 *(\mathrm{d}z\wedge\tilde{u}) \tag{6.2}$$

and we compute

$$i_{\partial_r}F^{\mathrm{I}}=\gamma^2 b_0(\omega r\sin^2\theta e^0-\sin\theta e^3) \tag{6.3}$$

$$i_{\partial_r}F^{\mathrm{II}}=\left(-\frac{\partial\Phi_{\mathrm{E}}^{\mathrm{II}}}{\partial r}\right)e^0+\left(\frac{1}{r}\frac{\partial\Phi_{\mathrm{M}}^{\mathrm{II}}}{\partial\theta}\right)e^3 \tag{6.4}$$

$$i_{\partial_r}\{\gamma m_0*(dz\wedge\tilde{u})\}=(\gamma^2 m_0\omega r\sin^2\theta)e^2-(\gamma^2 m_0\sin\theta)e^3. \tag{6.5}$$

Hence from (3.3) and (6.2) the discontinuity current 1-form is

$$j_\Delta=-\left\{\left[\varepsilon_0\left(\frac{\partial\Phi_{\mathrm{E}}^{\mathrm{II}}}{\partial r}\right)+\gamma^2\omega r\sin^2\theta(\varepsilon_0 b_0-m_0)\right]e_0\right.$$
$$\left.\left.+\left[\varepsilon_0\left(\frac{1}{r}\frac{\partial\Phi_{\mathrm{M}}^{\mathrm{II}}}{\partial\theta}-\gamma^2 b_0\sin\theta\right)+\gamma^2 m_0\sin\theta\right]e^3\right\}\right|_{r=a}. \tag{6.6}$$

We now define a surface charge density 2-form Σ_v and a surface current density 1-form κ_v by

$$*j_\Delta=\Sigma_v\wedge\mathrm{d}f+\kappa_v\wedge\mathrm{d}f\wedge\tilde{v} \tag{6.7}$$

where $v=\partial/\partial t$, $i_v\Sigma_v=0$, $i_v\kappa_v=0$ and $f\equiv r-a$ characterises the surface of the sphere. Then we read off the inertial surface charge density 2-form Σ_v and inertial surface current density 1-form κ_v by comparing (6.6) and (6.7):

$$\Sigma_v=\left.\left\{\varepsilon_0\left(\frac{\partial\Phi_{\mathrm{E}}^{\mathrm{II}}}{\partial r}\right)-(m_0-\varepsilon_0 b_0)\gamma^2\omega r\sin^2\theta\right\}e^2\wedge e^3\right|_{r=a} \tag{6.8}$$

$$\kappa_v=\left.\left\{\varepsilon_0\left(\frac{1}{r}\frac{\partial\Phi_{\mathrm{M}}^{\mathrm{II}}}{\partial\theta}-\gamma^2 b_0\sin\theta\right)+\gamma^2 m_0\sin\theta\right\}e^2\right|_{r=a}. \tag{6.9}$$

It is straightforward to verify that the total inertial surface charge

$$Q^{\mathrm{surf}}=\int_{S^2}\Sigma_v \tag{6.10}$$

is equal and opposite to the total inertial volume charge (4.17)

$$Q^{\mathrm{vol}}=\int_{S^2}*G^{\mathrm{I}}. \tag{6.11}$$

Furthermore we note that Q^{surf} and Q^{vol} separately approach zero as ω tends to zero. Since for $\gamma\sim 1$, $\beta_1=b_0a^3/2$ and $\beta_j=0$ (for $j\neq 1$), we have

$$\left.\frac{1}{r}\frac{\partial\Phi_{\mathrm{M}}^{\mathrm{II}}}{\partial\theta}\right|_{r=a}=-\tfrac{1}{2}b_0\sin\theta \tag{6.12}$$

and hence

$$\kappa_v|_{\omega=0} = -(\tfrac{3}{2}\varepsilon_0 b_0 - m_0) \sin\theta e^2 \tag{6.13}$$

Then $\kappa_v|_{\omega=0} = 0$ with the choice

$$b_0 = \tfrac{2}{3}\mu_0 m_0. \tag{6.14}$$

Thus (6.14) completes the specification of our solution in terms of a, ω and m_0.

7. Conclusions

We have presented the stationary electromagnetic field of a magnetised conducting sphere executing a rigid rotation. The inertial components of the exterior field have been given as a multipole series whilst the interior field can be specified simply in terms of a magnetisation strength and an arbitrary angular speed. The fields tend to the known [9] static magnetic fields of a non-rotating magnetised sphere and have the expected behaviour found by other authors in the limit of non-relativistic rotational speeds.

As emphasised in the introduction our approach has not relied on the use of any infinitesimal inertial Lorentz transformation [9] to generate the fields. Rather we have been led naturally to a simple magnetisation 2-form Π appropriate for the problem. Different choices of self-consistent magnetisations and induced interior currents (with the same static limits) may be contemplated. We consider the system described in this paper to be among the simplest. Others have been studied but with a considerable increase in complexity. Such complexity is commensurate with that produced by relaxing our axial symmetry condition and calculating the radiation produced when the axes of rotation and magnetisation no longer coincide. These effects may be found in [10]. Aside from the generation of higher-multipole exterior fields perhaps the most notable property of the above solution is the concentration of inertial (free) volume charge density in the equatorial plane ($\theta = \pi/2$) of the sphere as ω increases. This may be compared with the constant charge density for the interior of a slowly rotating sphere. There is a corresponding redistribution of surface charge in the relativistic case along with the appearance of surace 3-current. Such relativistic effects might have interesting implications for the behaviour of a magnetised rotating core with a (corotating) magnetosphere.

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